Math 210A Lecture 19 Notes

Daniel Raban

November 14, 2018

1 The Jordan-Hölder Theorem and Solvable Groups

1.1 The Jordan-Hölder theorem

Last time we had a lemma which said that if $N \trianglelefteq G$, then a composition series for N comes from a composition series for G by taking $H_i \cap N$ and eliminating duplicates. A composition series for G/N comes from $H_i N/N$ and eliminating duplicates. If the composition series for N has length r, and the composition series for G/N has length s, then $r + s = t$, where t is the length of the composition series for G.

Lemma 1.1. *Let $N \trianglelefteq G$. There exists a 1 to 1 correspondence between subgroups of G containing N and subgroups of G/N.*

Lemma 1.2. *Let $N \trianglelefteq G$ have composition series $1 = H_0 \trianglelefteq \cdots \trianglelefteq H_s = N$ and G/N have composition series $1 = Q_0 \trianglelefteq \cdots \trianglelefteq Q_r = G/N$. Then let H_{s+i} be the unique subgroup of G containing N with $N_{s+i}/N = Q_i$. Then $1 = H_0 \trianglelefteq \cdots \trianglelefteq H_t = G$ for $t = r + s$ is a composition series for G, and $H_{s+i}/H_{s+i-1} \cong Q_i/Q_{i-1}$.*

Theorem 1.1 (Jordan-Hölder)**.** *Let G be a finite group.*

1. *G has a composition series.*
2. *If $G \neq 1$ with two composition series $(K_i)_{i=0}^s$ and $(H_j)_{j=0}^t$, then $s = t$, and there exists $\sigma \in S_t$ such that $H_{\sigma(i)}/H_{\sigma(i)-1} \cong K_i/K_{i-1}$.*

Proof. Proceed by induction on $|G|$. If G is simple, $1 \trianglelefteq G$ is the only composition series, and we are done. If G is not simple, there there exists a proper normal subgroup $N \trianglelefteq G$ with $N \neq 1$. By induction, N and G/N have composition series. By the lemma, G has a composition series, as well.

To prove the second statement induct on the minimal length s of a composition series $(K)_{i=0}^s$. If $s = 1$, then G is simple, so this case is done. Let $N = K_{s-1} \trianglelefteq G$. N has the composition series $(K_i)_{i=0}^{s-1}$. N also has the composition series $(H_{f(i)} \cap N)_{i=0}^r$ where $f : \{0, \dots, r\} \to \{0, \dots, t\}$ is increasing with $f(0) = 0$. By induction, $r = s - 1$, and there exists a $\sigma \in S_{s-1}$ such that $K_i/K_{i-1} \cong (H_{f(\sigma(i))} \cap N)/(H_{f(\sigma(i))-1} \cap N)$.

Let $k < r$ be maximal such that $H_{k-1} \leq N$. Then $H_{k-1} \cap N = H_{k-1} \trianglelefteq H_k \cap N < H_k$. So $H_{k-1} = H_k \cap N$, which implies that $k \notin \operatorname{im}(f)$. Then $H_k/H_{k-1} \cong H_k/(H_k \cap N) \cong H_kN/N = G/N$. If $(H_iN)/(H_{i-1}N) \neq 1$ for $i \neq k$, then G/N has composition series of length ≥ 2, but G/N is simple. So $r = t - 1$. $\square$

1.2 Solvable groups

Definition 1.1. Let $G^{(i)}_{i \geq 0}$ be descending. The series $G^{(0)} = G$, $G^{(1)} = G' = [G, G]$, with general term $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for all $i \geq 0$ is called the **derived series** of G.

Definition 1.2. A group G is **solvable** if it has finite derived serires.

Example 1.1. Abelian groups are solvable.

Example 1.2. Semidirect products of abelian groups are solvable. If $G = N \rtimes H$, then $G' \leq N$ and $G'' = 1$.

Example 1.3. Simple nonabelian groups are not solvable. If G is simple and nonabelian, then $G' = G$.

Example 1.4. Let R be a commutative ring. The Heisenberg group

$$H = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \subseteq \mathrm{GL}_3(R)$$

is solvable.

$$\left[\begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so

$$H' = \left\{ \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} = Z(H),$$

and $H'' = 1$.

Proposition 1.1. *The following are equivalent:*

1. *G is solvable.*
2. *G has a normal series with abelian composition factors.*
3. *G has a subnormal series with abelian composition factors.*

Proof. We need only show that $3 \implies 1$. Let $1 = N_t \trianglelefteq \cdots \trianglelefteq N_1 \trianglelefteq N_0$ with abelian composition factors. Then G/N_i is abelian iff $G' \leq N_i$. N_{i-1}/N_i is abelian, so $N_i \geq (N_{i-1})' \geq G^{(i+1)}$. So $G^{(t)} = 1$ so G is solvable. $\square$

Lemma 1.3. *Let G be a group.*

1. *If G is solvable, then $H \leq G$ is solvable and G/N is solvable for $N \trianglelefteq G$.*
2. *If $N \trianglelefteq G$ and G/N are both solvable, then G is solvable.*

Proposition 1.2. *A group G with a composition series is solvable if and only if it is finite and its Jordan Hölder factors are all cyclic of prime order.*